Example

Compute the limit $\lim_{x\to\infty}\left(\sqrt{1+\frac{1}{x}}-1\right)x$

Solution: $\sqrt{1+a} = ?$ geometric is $a=-r, p=-1$.

Converges for $|a|<1$

$$(1+a)^p = 1 + pa + \frac{p(p-1)}{2!}a^2 + \frac{p(p-1)(p-2)}{3!}a^3 + \dots$$

$$\left(\sqrt{1+a}\right) = (1+a)^{1/2} = 1 + \frac{1}{2}a + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}a^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}a^3 + \dots$$

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{1+\frac{1}{x}}-1\right)x = \lim_{x\to\infty}\left(\left(1+\frac{1}{x}\right)^{1/2}-1\right)x$$

$\frac{1}{x} \to 0$

$$= \lim_{x\to\infty}\left(\cancel{1}+\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{1}{x}\right)^2+\dots-\cancel{1}\right)x$$

$$= \lim_{x\to\infty}\left(\frac{1}{2}\frac{1}{\cancel{x}}\cancel{x} - \frac{1}{4\cdot 2}\frac{1}{\cancel{x^2}_{x}}\cdot\cancel{x} + \dots\right) = \boxed{\frac{1}{2}}$$

$\to 0 \qquad \to 0$

Ex $\lim_{y\to 0}\dfrac{\left(e^{y^5}-1-y^4\sin(y)\right)}{y^{10}}$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$\sin(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!} + \dots$

$$= \lim_{y\to 0}\frac{\left(\left(\cancel{1}+\cancel{y^5}+\frac{(y^5)^2}{2!}+O(y^{15})\right)-\cancel{1}-y^4\left(\cancel{y}-\frac{y^3}{3!}+\frac{y^5}{5!}\right)\right)}{y^{10}}$$

order of y^{15}

$$= \lim_{y\to 0} \frac{\frac{y^{10}}{2} + O(y^{15}) + \frac{y^7}{3!} - \frac{y^9}{5!} + O(y^{11})}{y^{10}}$$

$$= \lim_{y\to 0} \frac{1}{2} + \underset{\substack{\downarrow \\ 0}}{O(y^5)} + \underset{\substack{\downarrow \\ \infty \\ \text{fast}}}{\frac{1}{6y^3}} - \underset{\substack{\downarrow \\ \infty \\ \text{slow}}}{\frac{1}{120y}} + \underset{\substack{\downarrow \\ 0}}{O(y)}$$

$$= \boxed{+\infty}$$

Series convergence / divergence

Rule of thumb: with factorials, use Ratio Test.
with n^{th} powers & no factorials, use Root Test.

Important example limit: $\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y = e$

$$\parallel$$

$$\lim_{y\to\infty}\left(\frac{y+1}{y}\right)^y = e$$

$\boxed{\text{Ex}}$ Does $\sum_{n=5}^{\infty} \frac{n3^n}{(n+1)!}$ converge?

Solution: Ratio Test: $b_n = \frac{n3^n}{(n+1)!}$, $b_{n+1} = \frac{(n+1)\cdot 3^{n+1}}{(n+2)!}$

$((n+1)+1)!$

$$L = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)3^{n+1}}{(n+2)!}\cdot\frac{(n+1)!}{n3^n}\right|$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)!}}{\underbrace{(n+2)!}_{(n+2)\cancel{(n+1)!}}} \underbrace{\frac{(n+1)}{n}}_{1} \cdot \frac{3^{n+1}}{3^n} = \lim_{n\to\infty} \frac{1}{(n+2)} \cdot 1 \cdot 3 = 0$$

$L = 0 < 1$

$\Rightarrow$ series converges by the ratio test!

$\boxed{\text{Example}}$ Does $\displaystyle\sum_{k=4}^{\infty} \frac{\sqrt{k^k}}{(k^2)(k-1)!}$ converge?

Let's use the ratio test: $b_k = \dfrac{\sqrt{k^k}}{(k^2)(k-1)!}$, $b_{k+1} = \dfrac{\sqrt{(k+1)^{k+1}}}{(k+1)^2\, k!}$

$$\lim_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right| = \lim_{k\to\infty} \frac{\sqrt{(k+1)^{k+1}}}{(k+1)^2\, k!} \cdot \frac{k^2 (k-1)!}{\sqrt{k^k}}$$

$$= \lim_{k\to\infty} \frac{\sqrt{(k+1)^{k+1}}}{\sqrt{k^k}} \cdot \frac{k^2}{(k+1)^2} \cdot \frac{(k-1)!}{k!}$$

$$= \lim_{k\to\infty} \sqrt{\frac{(k+1)^k (k+1)}{k^k}} \cdot \underbrace{\frac{k^2}{(k+1)^2}}_{1} \cdot \frac{1}{k}$$

$$= \lim_{k\to\infty} \sqrt{\underbrace{\left(\frac{k+1}{k}\right)^k}_{e}} \frac{\sqrt{k+1}}{k} = \lim_{k\to\infty} \sqrt{e} \cdot \sqrt{\frac{k+1}{k^2}}$$

$$= \lim_{k\to\infty} \sqrt{e} \cdot \underbrace{\sqrt{\frac{1}{k} + \frac{1}{k^2}}}_{0} = 0.$$

$L = 0 < 1 \Rightarrow$ series converges by ratio test.

Ex Does $\sum_{m=10}^{\infty} \frac{(3^m m^m + 1)(-1)^m}{(2m+3)^m 2^m}$ Converge?

Root test: $L = \lim_{m\to\infty} \sqrt[m]{|b_m|}$

$$= \lim_{m\to\infty} \sqrt[m]{\frac{3^m m^m + 1}{(2m+3)^m 2^m}} = \lim_{m\to\infty} \frac{(3^m m^m + 1)^{\frac{1}{m}}}{(2m+3)2}$$

$$= \lim_{m\to\infty} \frac{\left(3^m m^m\left(1+\frac{1}{3^m m^m}\right)\right)^{1/m}}{(2m+3)2}$$

$$= \lim_{m\to\infty} \frac{3m\left(1+\frac{1}{3^m m^m}\right)^{1/m}}{(2m+3)(2)} = \lim_{m\to\infty} \frac{3m}{(2m+3)2}$$

$\to 1$

$$= \lim_{m\to\infty} \frac{\cancel{m}(3)}{\cancel{m}(2+\frac{3}{m})2} = \frac{3}{4} = L \Rightarrow L < 1$$

$\downarrow 0$

By root test, the series converges.

Big use of Ratio & Root tests — finding intervals of convergence of Taylor series.

$T_\infty(x) = \sum a_n (x-a)^n \to$ ratio or root test $L < 1$

$\downarrow$

After simplifying $|x-a| < (\#)$

converges if $|x-a| < (\#)$.